Splitting the voter Potts model critical point

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Recently some two-dimensional models with double symmetric absorbing states were shown to share the same critical behavior that was called the voter universality class. We show that, for an absorbing-states Potts model with finite but further than nearest-neighbor range of interactions, the critical point is split into two critical points: one of the Ising type and the other of the directed percolation universality class. Similar splitting takes place in the three-dimensional nearest-neighbor model.

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Nonequilibrium phase transitions are recently attracting increasing theoretical interest. One of the motivations driving the research in this field is the idea that, similar to equilibrium systems, nonequilibrium continuous phase transitions can be also divided into several, hopefully not so many, universality classes [1]. Indeed, there are some examples that show that such a classification, at least to some extent, can be made. Particularly interesting results exist for models with absorbing states, and a notable example is the directed percolation (DP) universality class. A conjecture by Jenssen and by Grassberger [2] that models with a single absorbing state should belong to the same universality class has by now very convincing supports. Another universality class encompasses models with double absorbing states [3,4], which in one dimension ($d=1$) includes also models whose dynamics has some additional symmetry (parity conservation) [5]. However, it is also known that any asymmetry in the dynamics, which would favor any of the absorbing states, drives the system into a DP universality class [6]. Universality of models with more than two absorbing states is even more problematic, because for models with symmetric absorbing states some details of the dynamics might affect the critical behavior too [7]. It is becoming clear that a task of classifying nonequilibrium phase transitions is far from completed [8].

Numerical results that support the above classification come mainly from one-dimensional models. The situation for $d>1$ is less understood, but there are some results in this case too. In addition to numerous examples of models with a single absorbing state, and thus belonging to DP universality class [1], there are some indications that for $d>1$ models with more than two symmetric absorbing states the phase transition should be discontinuous [9]. In between, there are models with two absorbing states. Dornic *et al.* have shown that in this case and for $d=2$ a group of models belongs to a new universality class called a voter universality class [10]. Originally, a voter model was introduced in a rather nonphysical context of opinion spreading [11]. Later on numerous examples of related and physically more relevant models were also studied [12]. Although the phase transition in the voter universality class is continuous, the decay of the order parameter ρ upon approaching the critical point is slower than any power-law decay $\rho\sim\epsilon^{\beta}$, where ϵ measures the distance from the critical point. Formally, such a decay might be described with $\beta=0$. In addition, the time decay of the order parameter at criticality is also slower than any power-law decay and is, in fact, logarithmic [$\rho\sim 1/\ln(t)$] as shown exactly [13]. Some other exponents of voter universality class were also determined [10].

An interesting feature of the voter model, which was not yet addressed, is the fact that at its critical point actually two phenomena seem to take place. One of them is the symmetry breaking between two competing states of the model, which is similar to the symmetry breaking in the equilibrium Ising model. The second phenomenon is the phase transition between active and absorbing phases of the model. Since there is no symmetry breaking transition in one-dimensional equilibrium Ising model, it is easy to understand that for models with two absorbing states in $d=1$, the symmetry breaking must result from the absorbing phase transition and thus both phenomena take place simultaneously. But this is no longer the case for $d>1$ models, and the coincidence of these two transitions should not be taken for granted.

In the present paper we study a recently introduced nonequilibrium Potts model whose dynamics has two absorbing states [9]. We show that the model with nearest-neighbor interactions on square lattice belongs to the voter universality class. However, with an extended range of interactions (up to third nearest neighbors) the voter critical point is split. Starting from the disordered phase and reducing a temperaturelike control parameter T, the model first undergoes symmetry breaking phase transition. Calculation of the Binder cumulant indicates that this transition belongs to the Ising universality class. Upon further decrease in T, the model undergoes a second phase transition into the absorbing phase. Since at this point the symmetry is already broken, this second transition, as expected, belongs to the DP universality class. A similar behavior is observed for the nearest-neighbor model in the three-dimensional case.

The observation that a certain type of a nonequilibrium critical point can be considered as superposition of two other critical points is, in our opinion, new, and hopefully, it should increase the understanding of nonequilibrium phase transitions. Let us notice that a superposition of different critical points exists in some equilibrium systems. For example, in the frustrated XY model two phase transitions of the Kosterlitz-Thouless type [$U(1)$] and of the Ising type

(Z_2) under certain conditions most likely happen simultaneously [14]. Other examples are multicritical points in random magnets [15] or in a diffusive kinetic Ising model [16].

Before defining our model, let us recall that the equilibrium ferromagnetic two-state Potts model can be defined using the following Hamiltonian [17]:

$$H=-\sum_{(i,j)} \delta_{\sigma_i\sigma_j}, \tag{1}$$

where summation is over pairs (i,j) of interacting sites on a Cartesian lattice of the linear size L. With each site i we assign a variable $\sigma_i=0,1$ and δ is the Kronecker delta function.

To study model (1) using Monte Carlo simulations, one constructs a stochastic Markov process with suitably chosen transition rates. One of the possible choices corresponds to the so-called Metropolis algorithm. In this algorithm one looks at the energy difference ΔE between the final and initial configurations and accepts the move with probability $\min\{1,e^{-\Delta E/T}\}$, where T is temperature. To obtain a model that would have symmetric absorbing states, we modify the Metropolis algorithm of model (1) as follows [9]: when all neighbors of a given site i are in the same state as this site, then site i cannot change its state. Thus, the dynamics of our model is defined as follows: (i) Select randomly the site i and its possible final state. (ii) If $\Delta E<z$, update the site i with the probability $\min\{1,e^{-\Delta E/T}\}$, where z is the number of neighbors interacting with site i (in our paper z is i independent). Let us notice, that after such a modification, T is no longer temperature. Nevertheless, we will keep such a terminology. Moreover, the unit of time is defined as a single (on average) update of every site.

We used numerical simulations to examine the properties of our model. A natural characteristic of models with absorbing states is the steady-state density of active sites ρ. In our model, a given site i is active when at least one of its neighbors is in a state different than i. Otherwise the site i is called nonactive. Moreover, we used the so-called dynamic Monte Carlo method where one sets the system in the absorbing state with activity only locally initiated and measure some stochastic properties of runs [18]. Typical characteristics are the survival probability $P(t)$ that activity survives at least until time t and the average number of active sites $N(t)$ (averaged over all runs). At criticality $P(t)$ and $N(t)$ are expected to have power-law decay: $P(t)\sim t^{-\delta}$ and $N(t)\sim t^{\eta}$. To detect a possible symmetry breaking in the model, we measured the magnetization $m=2/L^d\langle\Sigma_i\sigma_i\rangle-1$ and its variance $s=1/L^d\langle(2\Sigma_i(\sigma_i-\langle\sigma_i\rangle)-L^d)^2\rangle$. At the symmetry breaking critical point the variance s, which is related to magnetic susceptibility for equilibrium systems, is expected to diverge in the limit $L\to\infty$. Below we present the results of our simulations. We ensured that the lattice size L is large enough for the finite-size effects to be negligible.

Model with nearest-neighbor interactions on square lattice. In this case the model was already studied by some of us, but only in the context of an absorbing phase transition [9]. A very slow decay of ρ at the critical point was observed, but being unaware of the relation with the voter model, we could not draw definite conclusions about the nature of the critical point in this case. Indeed, having a double symmetric absorbing state and being driven only by interfacial noise, the model satisfies the criterions of Dornic et al. for belonging to the voter universality class. Additional confirmation is shown in Fig. 1. One can see that the variance s diverges at the same temperature where ρ vanishes. Moreover, s^{-1} seems to decay linearly at the critical point, which implies that $\gamma=1$, as also predicted for this universality class [10].

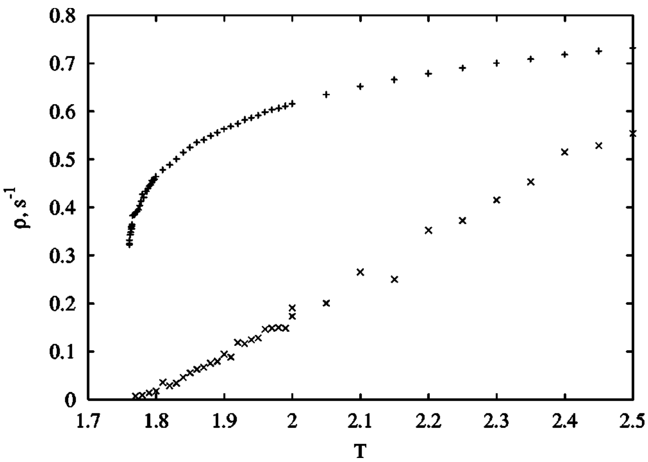

FIG. 1. The density of active sites ρ (+) and the inverse of the variance of magnetization s multiplied by a factor of 5 ($\times$) as a function of temperature for the two-dimensional nearest-neighbor model. Simulations were done for $L=500$. Close to the critical temperature T_c (=1.7585 [9]) the linear decay of s^{-1} might be affected by some logarithmic corrections [10].

Model with up to third-nearest-neighbor interactions on square lattice. Studying our nonequilibrium Potts model within a mean-field approximation at the pair level, we noticed that, for sufficiently large coordination number z, the structure of the solution qualitatively changes in a way that clearly indicates two separate transitions in the model [19]. It prompted us to simulate our model with interactions including also further neighbors. First we studied the model with interactions up to second-nearest neighbors $(z=8)$. It turned out, that in this case either the model belongs to the voter universality class or there is only extremely small splitting beyond the resolution of our simulations.

However, for the model with interactions up to third-nearest neighbors $(z=12)$, a qualitatively new picture emerges. Indeed, one can see in Fig. 2 that the variance s diverges at temperature T_I, which is clearly larger than temperature T_c, where ρ vanishes. In the temperature interval $T_c<T<T_I$ our model is magnetized (Fig. 2). To avoid rather slow coarsening effects in this interval, it is better to start simulations from an asymmetric $(m\neq 0)$ initial configuration.

To examine the nature of the phase transition at $T=T_I$, we calculated the so-called Binder cumulant [20] $U=1-m_4/3m_2^2$, where m_n is the nth moment of magnetization. In Fig. 3 one can see that at the crossing point U is relatively close to the universal value $U=0.6107$ of the $d=2$ Ising model [21]. Although our model is nonequilibrium type, its

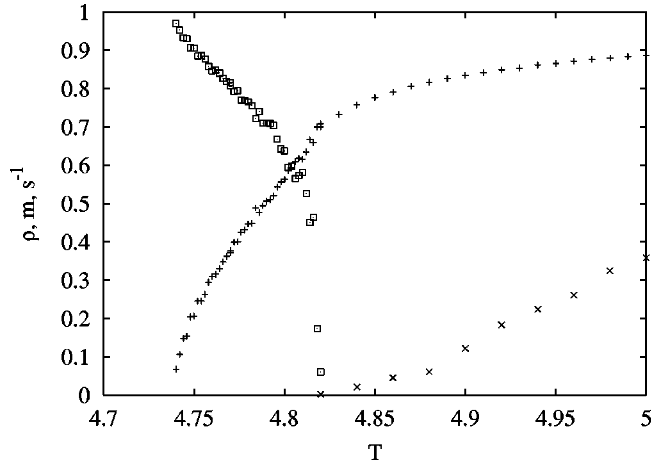

FIG. 2. The density of active sites ρ (+), magnetization m (□), and the inverse of the variance of magnetization s multiplied by a factor of 30 (×) as a function of temperature for the two-dimensional model with 12 neighbors ($L=400$).

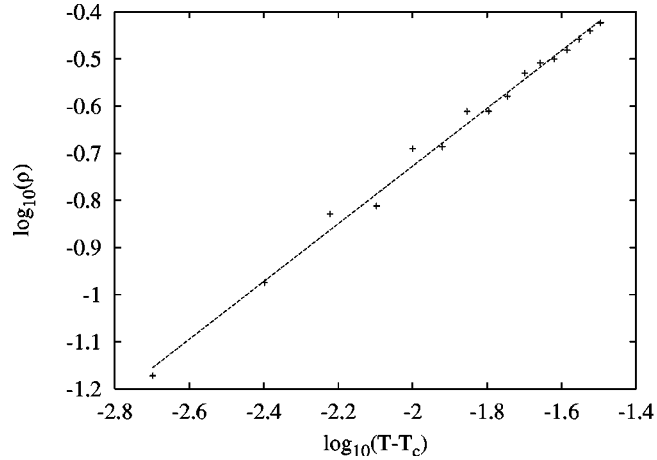

FIG. 4. The scaling of the density of active sites ρ in the vicinity of the critical point $T_c=4.738$ in the two-dimensional model with 12 neighbors. Simulations were done for $L=500$ and the linear fit has a slope $\beta=0.61$.

critical behavior at $T=T_I$ is the same as in equilibrium (Ising) systems. Such a feature is in agreement with some expectations [22] that at the critical point of many nonequilibrium systems only some general properties (e.g., symmetry) determine the nature of the critical point while some others, as e.g., a lack of detailed balance, are very often irrelevant in this respect. Since the symmetry is already broken upon approaching the critical point at $T=T_c$, we expect that this critical point should belong to the (2+1) DP universality class. Simulations confirm these expectations. In Fig. 4 one can see that close to the critical point at $T=T_c$ the density of active sites behaves as $\rho\sim(T-T_c)^{\beta}$ and we estimate that $\beta=0.61(4)$, which can be compared with the DP value 0.584 [1]. Additional confirmation is obtained using the dynamical Monte Carlo method, which enables us also to precisely locate the critical temperature $T_c=4.7380(5)$. From the measurement of $N(t)$ (Fig. 5) at criticality we estimate $\eta=0.25(3)$, which is in a reasonable agreement with DP value 0.230 [1]. We also measured $P(t)$ and from these data (not presented here) we estimate $\delta=0.44(3)$, which can be compared with the DP value 0.451 [1]. Our estimation of dynamical exponents is much different from that obtained for some models of voter universality class $\delta\sim0.9$, $\eta\sim0.0$ [4,9].

Let us note that similar to the nearest-neighbor case, the $z=12$ model also has two absorbing states and is driven only by interfacial noise. Nevertheless, the critical behavior in this case is much different from the voter model. This is thus yet another example that shows that simple criterions most likely cannot be used for classifying nonequilibrium critical points.

Model with nearest-neighbor interactions on simple cubic lattice. We also studied the three-dimensional nearest-neighbor version of our model ($z=6$). Similar to the z

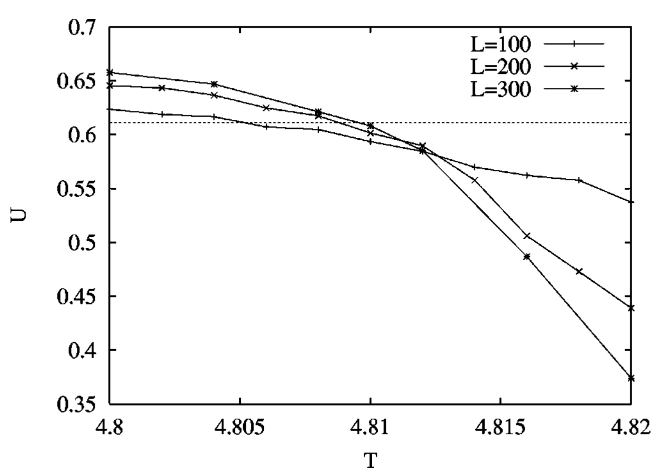

FIG. 3. The Binder cumulant U as a function of temperature T for the two dimensional model with 12 neighbors. The horizontal dotted line denotes the universal value $U\sim0.6107$ for the $d=2$ Ising model. To diminish fluctuations, long simulations were made with the simulation time $t\sim5\times10^6$ Monte Carlo steps.

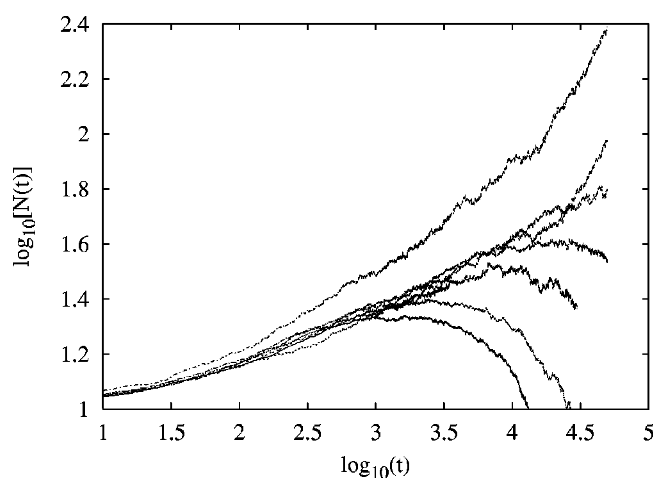

FIG. 5. The average of the number of active sites $N(t)$ as a function of time t calculated using dynamical Monte Carlo for the two-dimensional next-next-nearest-neighbor model and (from top) $T=4.739$, 4.7385, 4.738 (critical point), 4.7375, 4.737, 4.736, 4.735. For each temperature, the average is made usually over 2×10^4 independent runs ($L=5000$).

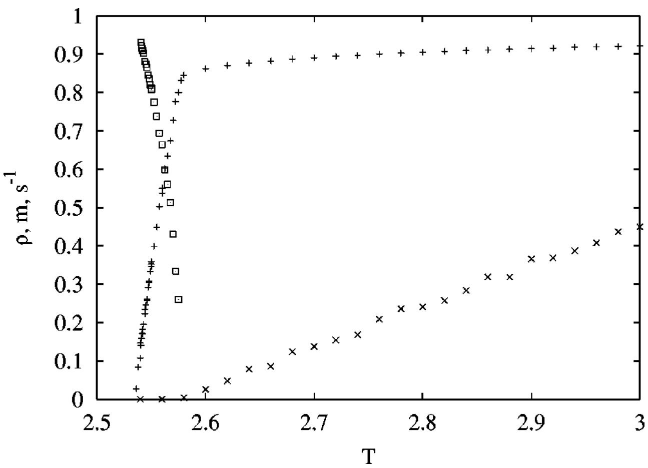

FIG. 6. The density of active sites ρ (+), magnetization m (□), and the inverse of the variance of magnetization s multiplied by a factor of 5 (×) as a function of temperature for the three-dimensional nearest-neighbor model ($L=60$).

$=12$ case, here again the absorbing and magnetic phase transitions are separated (Fig. 6). We expect that in this case the magnetic phase transition belongs to the $d=3$ Ising universality class and the absorbing phase transition belongs to the $(3+1)$ DP universality class. Confirmation of such a scenario will require, however, extensive numerical simulations and is left for the future.

In summary, we have shown that models with two absorbing states in $d>1$ dimensions might exhibit two transitions where the first one breaks the symmetry and the second one brings the model into an absorbing state. In the voter model and some related models, these two transitions coincide. Hopefully, such an interpretation of the voter model will contribute to a better understanding of its unusual critical behavior. For example, diminishing the strength of the further-neighbor interactions, we can reduce the splitting T_I-T_c and examine a crossover to the voter universality class at which $T_I=T_c$. Of course, the Ising-type phase transition is not the only type of the symmetry breaking and other types, e.g., three-state Potts or XY, are also possible in nonequilibrium systems. It would be interesting to examine whether such critical points can be superposed with a DP universality class, which might result in new types of critical behaviors.

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